Lecture 25.

Thm 1. Let $T \in GL(n, \mathbb{R})$.

(i) $E \in \mathcal{L}^n \Rightarrow T(E) \in \mathcal{L}^n$ and

$$m(T(E)) = |\det T|\, m(E).$$

(ii) f $\mathcal{L}^n$-meas. $\Rightarrow f \circ T$ $\mathcal{L}^n$-meas.

If $f \in L^+$ or L^1, then

$$\int f\,dm = |\det T| \int (f \circ T)\,dm$$

Pf. We first recall that any $T \in GL(n, \mathbb{R})$ can be written as $T = S_1 \circ S_2 \circ \cdots \circ S_m$, where the S_j are elementary matrices, i.e. of the forms:

(I) $S = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \Rightarrow \det S = c$

(II) $S = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} \Rightarrow \det S = -1$

(III) $S = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & \cdots c & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \Rightarrow \det S = 1$

For pf of Thm 1, first establish (ii) for f $\mathcal{B}_{\mathbb{R}^n}$-meas. and elementary matrices T.

Since T is cont., $f \circ T$ is $\mathcal{B}_{\mathbb{R}^n}$-meas.

We check integral when $f \in L^+$ or L^1 for each elementary type of T:

(I) Using FT we get

$$\int (f \circ T)\, dm = \int \cdots \int \left(\int f(cx_j)\, dx_j \right) \widehat{dx_j}$$

$dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_n$

$$= |c|^{-1} \int f\, dm = |\det T|^{-1} \int f\, dm$$

Conseq. of Thm 1.21 in 1-D as in pf of Thm 1.

$$\begin{cases} m^1(rE) = |r|\, m^1(E) \\ m^1(a+E) = m^1(E). \end{cases}$$

(II) $\int (f \circ T)\, dm = \int f\, dm$ by FT (swap order $i \leftrightarrow j$).

and $|\det T| = |-1| = 1$

(III) $\int (f\circ T)\,dm = \int \cdots \int \Big(\underbrace{\int f(x_j + c x_i)\,dx_j}_{\substack{= \int f(x_j)\,dx_j \\ \text{by Thm 1.21}}} \Big)\, \widehat{dx_j}$

$= \int f\,dm$, and $|\det T| = 1$.

Now, for any $T = S_1 \circ \cdots \circ S_m$, S_j elem.,

$\int f\,dm = |\det S_1| \int f\circ S_1\,dm = \ldots$

$|\underbrace{\det S_1 \cdots \det S_m}_{\det S_1 \circ \cdots \circ S_m}| \int f\circ S_1 \circ \cdots \circ S_m\,dm$

$= |\det T| \int (f\circ T)\,dm$.

Thus, we have established (ii) for f $\mathcal{B}_{\mathbb{R}^n}$-meas. $\Rightarrow$ (i) for $E \in \mathcal{B}_{\mathbb{R}^n}$

$\Big(\chi_E(Tx) = \chi_{T^{-1}E}(x) \Big)$

In particular, $m(N)=0 \Leftrightarrow m(TN)=0$ for all $N \in \mathcal{B}_{\mathbb{R}}$. The extension of (i) to all $E \in \mathcal{L}^n$ follows as in pf of Thm 1. $\Rightarrow$ (i) holds for $f = \chi_E$ all $E \in \mathcal{L}^n \Rightarrow$ for all $f = \varphi$ simple $\mathcal{L}^n$-meas. MCT + DCT $\Rightarrow \forall f$ as in Thm 1. $\boxtimes$

Change of variables.

Let $\Omega \subseteq \mathbb{R}^n$ open, $F: \Omega \to \mathbb{R}^n$ a $\mathcal{C}^1$-map ($\frac{\partial F}{\partial x_j} = F_{x_j}$ exist + cont.). Let DF_x denote linear map $\mathbb{R}^n \to \mathbb{R}^n$ given by matrix of partial derivatives at x. If $\det DF_x \neq 0$, $\forall x \in \Omega$, then by Inverse Function Thm, $F(\Omega) \subseteq \mathbb{R}^n$ is open.

<u>Def.</u> F is a <u>diffeomorphism</u> $\Omega \to \Omega'$ if $\Omega' = F(\Omega)$ and F is invertible with F^{-1} being $\mathcal{C}^1$.

<u>Rem.</u> By IFT, suffices that $\det DF_x \neq 0$ and F injective.

<u>Thm 2</u>. Let $\Omega \subseteq \mathbb{R}^n$ be open, $F: \Omega \to \Omega' = F(\Omega)$ a diffeomorphism.

(i) $E \in \mathcal{L}^n \Rightarrow F(E) \in \mathcal{L}^n$ and

$$m(F(E)) = \int_E |\det DF| \, dm$$

(ii) f $\mathcal{L}^n$-meas. on $\Omega' \Rightarrow f \circ F$ is $\mathcal{L}^n$-meas. on Ω. If $f \in L^+$ or L^1 then

$$\int_{\Omega'} f \, dm = \int_{\Omega} (f \circ F) |\det DF| \, dm$$

<u>Pf.</u> By similar arguments to the pf of Thm 1, suffices to prove the result for $\mathcal{B}_{\mathbb{R}^n}$ and $\mathcal{B}_{\mathbb{R}^n}$-meas. f. Also, (i) $\Rightarrow$ (ii) so suffices to prove (ii).

We shall need the following:

<u>Lemma 1</u>. Let $\Omega \subseteq \mathbb{R}^n$ open. Then $\exists$ $\{R_k\}_{k=1}^{\infty}$ closed equilateral cubes R_k w/ disjoint interiors, s.t. $\Omega = \bigcup_{k=1}^{\infty} R_k$.

Pf. Construction using $\underline{A}(\Omega)$ from Lecture 23